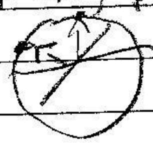
Michael Drake
8/2/10

homework question

sphere, pick arbitrary point, make it north pole and get equator



pick 10 north poles pick set of points
show that all points are $\perp$ to all north poles

for homework, increase radius of sphere

phase transitions

$G(n,p)$

as long as property is monotone, there's a phase transition for p, such that less than p doesn't have property and greater than p does have property.

when does graph have diameter 2 $p = \sqrt{2}\sqrt{\frac{\ln n}{n}}$

$p = c\sqrt{\frac{\ln n}{n}}$ $c > \sqrt{2}$ diameter is 2

if $\exists$ $p(n)$ such that $\lim\limits_{n\to\infty} \frac{p_1(n)}{p(n)} = 0$

then $G(n, p_1(n))$ doesn't have property

$\lim\limits_{n\to\infty} \frac{p_2(n)}{p(n)} = \infty$ then $G(n, p_2(n))$ has property

$p(n)$ is threshold and there's phase transition

often we have a random variable x that indicates how many copies of some item $G(n,p)$ has.

# of cycles. $\lim\limits_{n\to\infty} E(X) = \begin{cases} 0 & \text{Prob(graph has item)} = 0 \\ \infty & \text{prob(~~graph~~ has item)} > 0 \text{ NO!} \end{cases}$

all graphs 90% 0 cycles | 10% 1 or more cycles   0 90% | ∞ 10%

$E(X) = .1$

second moment method is used to show that when the expected value of a non-negative random variable is large compared to its variance then random variable takes on value 0 with probability 0

$\text{prob}(X=0) \le \text{Prob}(|X - E(X)| \ge E(X))$

by chebyshev's inequality $\left(\text{Prob}(|X - E(x)| \ge a\sigma) \le \frac{1}{a^2}\right)$

$\text{prob}(X=0) \le \text{Prob}(|X - E(X)| \ge E(X)) \le \frac{\sigma^2}{E^2(X)}$ $E(X) = a\sigma$ $a = \frac{E(X)}{\sigma}$

can claim so prob(graph has item) > 0 if $\lim \frac{\sigma^2}{E^2(X)} = 0$

$E(X)=0$ 0 2 2% $E=.01$

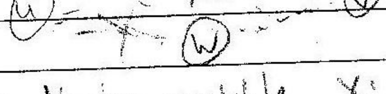
$E(X)=\infty$ 0 ∞ 1%

$\lim_{n\to\infty} \frac{\sigma^2}{E^2(X)}=0$ rules out the above behavior

threshold for diameter 2 $\quad p=c\sqrt{\frac{\ln n}{n}} \quad c=\sqrt{2}$

theorem: let $p=c\sqrt{\frac{\ln n}{n}}$ for $c<\sqrt{2}$ $G(n,p)$ almost surely has diameter greater than 2. $c>\sqrt{2}$ $G(n,p)$ almost surely has diameter less than or equal to 2

Proof: If G has diameter >2 $\exists$ vertices u and v that are not adjacent and no other vertex w is adjacent to both



u and v bad pair

indicator variable X_{ij} $\quad X_{ij}=\begin{cases}0 \\ 1 & \text{if } i \text{ and } j \text{ are bad pair}\end{cases}$

$X=\sum_{\substack{i,j \\ i\neq j}} X_{ij}$

prob w adj to both u and v $=p^2$

not adjacent to both $=1-p^2$

no vertex adj to both $=(1-p^2)^{n-2}$

u,v not adj $=1-p$ if n really large $e^{-c^2\ln n}$

$E(X)=\binom{n}{2}(1-p)(1-p^2)^{n-\cancel{2}} \quad p=c\sqrt{\frac{\ln n}{n}}$

$=n^2\left(1-c\sqrt{\frac{\ln n}{n}}\right)\left(1-c^2\frac{\ln n}{n}\right)^n$

$\lim_{n\to\infty} 1-0=1$ so now $n^2\left(1-c^2\frac{\ln n}{n}\right)^n$ second moment argument

$n^2e^{-c^2\ln n}=n^2n^{-c^2}=n^{2-c^2}=\begin{cases}c<\sqrt{2} & \infty \\ c>\sqrt{2} & 0\end{cases}$ no bad pairs graph diameter ≤ 2

$\frac{\sigma^2}{E^2(X)}$

$E(X^2)=E\left(\left(\sum_{ij}X_{ij}\right)^2\right)=E\left(\left(\sum_{ij}X_{ij}\right)\left(\sum_{kl}X_{kl}\right)\right)=E\left(\sum_{ijkl}X_{ij}X_{kl}\right)$

$\frac{\sigma^2}{E^2(X)}=\frac{E(X^2)}{E^2(X)}-\frac{E^2(X)}{E^2(X)}=\frac{E(X^2)}{E^2(X)}-1$

n^4 terms but if i and j are distinct from $k\,l$, then two variables are statistically independent

then $\sum_{ij\neq kl}\left(E(X_{ij}X_{kl})\right)$ $\quad E(X_{ij}X_{kl})=E(X_{ij})E(X_{kl})=0$

$E(X_{ij})=n^{-c^2}=0$